10
Number Systems

ENGR 3410 – Computer Architecture

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Decimal (Base 10) Numbers

- Positional system - each digit position has a value

$$2534 = 2*1000 + 5*100 + 3*10 + 4*1$$

- Alternate view: Digit position i from the right = Digit * 10^i
  (rightmost is position 0)

$$2534 = 2*10^3 + 5*10^2 + 3*10^1 + 4*10^0$$

Base R Numbers

- Each digit in range 0..(R-1)

  0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F ...

  A = 10

  B = 11

  C = 12

  D = 13

  E = 14

  F = 15

- Digit position i = Digit * R^i

  $D_3\ D_2\ D_1\ D_0$ (base R) = $D_3{*}R^3+D_2{*}R^2+D_1{*}R^1+D_0{*}R^0$

Conversion to Decimal

- Binary: $(101110)_2$

$$1\times2^5+0\times2^4+1\times2^3+1\times2^2+1\times2^1+0\times2^0$$
$$32+8+4+2=(46)_{10}$$

- Octal: $(325)_8$

$$3\times8^2+2\times8^1+5\times8^0$$
$$192+16+5=213_{10}$$

- Hexadecimal: $(E32)_{16}$

$$14\times16^2+3\times16+2$$
$$3584+48+2=3634_{10}$$

Conversion Decimal

- Binary: $(110101)_2$

- Octal: $(524)_8$

- Hexadecimal: $(A6)_{16}$

Conversion of Decimal to Binary (Method 1)

- For positive, unsigned numbers
- Successively subtract the greatest power of two less than the number from the value. Put a 1 in the corresponding digit position

- $2^0=1$ $2^4=16$ $2^8=256$ $2^{12}=4096$ (4K)
- $2^1=2$ $2^5=32$ $2^9=512$ $2^{13}=8192$ (8K)
- $2^2=4$ $2^6=64$ $2^{10}=1024$ (1K)
- $2^3=8$ $2^7=128$ $2^{11}=2048$ (2K)

Decimal to Binary Method 1

- Convert $(2578)_{10}$ to binary

$$\begin{array}{r} 2578_{10} \\ -\ 2048_{10} \\ \hline 530_{10} \end{array} \quad = 2^{11} \qquad \begin{array}{r} 530 \\ 512 \\ \hline 18 \end{array} \quad = 2^{9} \qquad \begin{array}{r} 18 \\ -\ 16 \\ \hline 2 \end{array} \quad = 2^{4} \qquad \begin{array}{r} 2 \\ -\ 2 \\ \hline 0 \end{array} \quad 2^{1}$$

$$(101000010010)_2$$

- Convert $(289)_{10}$ to binary

$$\begin{array}{r} 289 \\ 256 \\ \hline 33 \\ -\ 32 \\ \hline 1 \\ -\ 1 \\ \hline \end{array} \qquad \begin{array}{l} = 2^{8} \\ \\ = 2^{5} \\ \\ = 2^{0} \end{array}$$

$$(100100001)_2$$

Conversion of Decimal to Binary (Method 2)

- For positive, unsigned numbers
- Repeatedly divide number by 2. Remainder becomes the binary digits (right to left)
- Explanation:

Decimal to Binary Method 2

- Convert $(289)_{10}$ to binary

289	
144	1
72	0
36	0
18	0
9	0
4	1
2	0
1	0
0	1

$(100100001)_2$

Decimal to Binary Method 2

- Convert $(85)_{10}$ to binary

42	1
21	0
10	1
5	0
2	1
1	0
0	1

$(1010101)_2$

Converting Binary to Hexadecimal

- 1 hex digit = 4 binary digits
- Convert $(11100011010111010011)_2$ to hex

$(E\,3\,5\,D\,3)_{16} = 0xE35D3$

- Convert $(A3FF2A)_{16}$ to binary

$(1010,\ 0011,\ 1111,\ 1111,\ 0010,\ 1010)_2$

Converting Binary to Octal

- 1 octal digit = 3 binary digits
- Convert $(10100101001101010011)_2$ to octal

- Convert $(723642)_8$ to binary

Converting Decimal to Octal/Hex

- Convert to binary, then to other base
- Convert $(198)_{10}$ to Hexadecimal

- Convert $(1983020)_{10}$ to Octal

Arithmetic Operations

Decimal:

$$\begin{array}{r} 1\;1\;1 \\ 5\;7\;8\;9\;2 \\ +\;7\;8\;9\;5\;6 \\ \hline 1\;3\;6\;8\;4\;8 \end{array}$$

Binary:

$$\begin{array}{r} 1\;1\;1 \\ 1\;0\;1\;0\;1\;1\;1 \\ +\;0\;1\;0\;0\;1\;0\;1 \\ \hline 1\;1\;1\;1\;1\;0\;0 \end{array}$$

$2 = 10_2$

$3 = 11_2$

Decimal:

$$\begin{array}{r} 6\;8 \\ 5\;\cancel{7}\;8\;\cancel{9}\;{}^{1}2 \\ -\;3\;2\;9\;4\;6 \\ \hline 2\;4\;9\;4\;6 \end{array}$$

Binary:

$$\begin{array}{r} 0\;\;1\;\;10\;\;1\;\;1\;\;10\;\;10 \\ \cancel{1}\;\cancel{0}\;\cancel{1}\;\cancel{0}\;\cancel{0}\;\cancel{1}\;\cancel{1}\;{}^{1}0 \\ -\;0\;0\;1\;1\;0\;1\;1\;1 \\ \hline 0\;1\;1\;0\;1\;1\;1\;1 \end{array}$$

$d_n R^n$

Arithmetic Operations (cont.)

Binary:

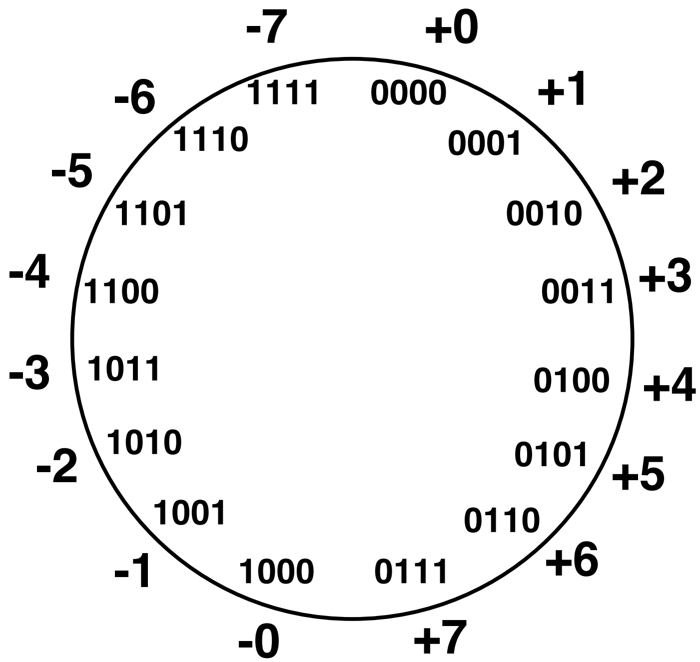
$$
\begin{array}{r}
1\ 0\ 0\ 1 \\
*\ \ 1\ 0\ 1\ 1 \\
\hline
1\ 0\ 0\ 1 \\
1\ 0\ 0\ 1\ \ \ \\
0\ 0\ 0\ 0\ \ \ \ \ \ \\
1\ 0\ 0\ 1\ \ \ \ \ \ \ \ \ \\
\hline
(1\ 1\ 0\ 0\ 0\ 1\ 1)_2
\end{array}
$$

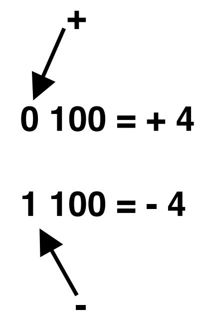
Negative Numbers

- Need an efficient way to represent negative numbers in binary
  - Both positive & negative numbers will be strings of bits
  - Use fixed-width formats (4-bit, 16-bit, etc.)

- Must provide efficient mathematical operations
  - Addition & subtraction with potentially mixed signs
  - Negation (multiply by -1)

Sign/Magnitude Representation





High order bit is sign: 0 = positive (or zero), 1 = negative

Three low order bits is the magnitude: 0 (000) thru 7 (111)

Number range for n bits = $+/-2^{n-1}-1$

Representations for 0:

Sign/Magnitude Addition

Idea: Pick negatives so that addition/subtraction works

$$\begin{array}{rrrrrl} & 0 & 0 & 1 & 0 & (+2) \\ + & 0 & 1 & 0 & 0 & (+4) \\ \hline & 0 & 1 & 1 & 0 & = +6 \end{array}$$

$$\begin{array}{rrrrrrl} & & 1 & 0 & 1 & 0 & (-2) \\ + & & 1 & 1 & 0 & 0 & (-4) \\ \hline & 1 & 0 & 1 & 1 & 0 & \end{array}$$

Low = +6 → 5 bits

High = −3

$$\begin{array}{rrrrrl} & 0 & 0 & 1 & 0 & (+2) \\ + & 1 & 1 & 0 & 0 & (-4) \\ \hline & 1 & 1 & 1 & 0 & = -6 \end{array}$$

$$\begin{array}{rrrrrl} & 1 & 0 & 1 & 0 & (-2) \\ + & 0 & 1 & 0 & 0 & (+4) \\ \hline & 1 & 1 & 1 & 0 & = -6 \end{array}$$

Bottom line: Basic mathematics are too complex in Sign/Magnitude

Idea: Pick negatives so that addition works

- Let -1 = 0 - (+1):

$$
\begin{array}{rrrrrl}
 & 0 & 0 & 0 & 0 & (\ 0) \\
- & 0 & 0 & 0 & 1 & (+1) \\
\hline
 & 1 & 1 & 1 & 1 & (-1)_{10}
\end{array}
$$

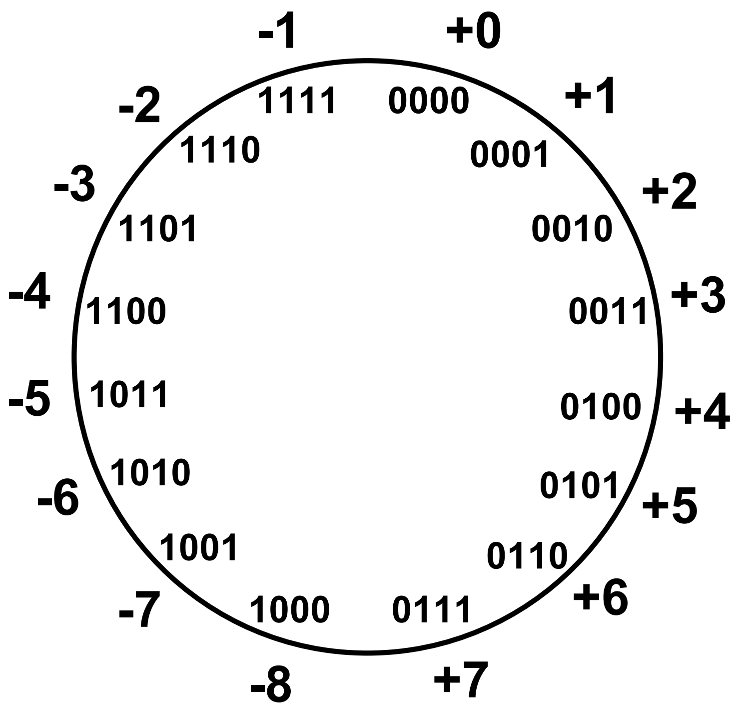
- Does addition work?

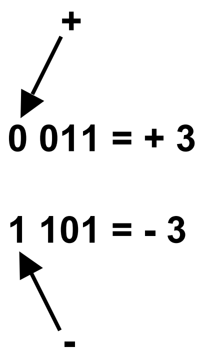
$$
\begin{array}{rrrrrrl}
 & & 1 & 1 & & & \\
 & & 0 & 0 & 1 & 0 & (+2) \\
+ & & 1 & 1 & 1 & 1 & (\ -1) \\
\hline
 & 1 & 0 & 0 & 0 & 1 &
\end{array}
$$

- Result: Two's Complement Numbers

Two's Complement

- Only one representation for 0
- One more negative number than positive number
- Fixed width format for both pos. & neg. numbers





Negating in Two's Complement

- Flip bits & Add 1
- Negate $(0010)_2$ (+2)

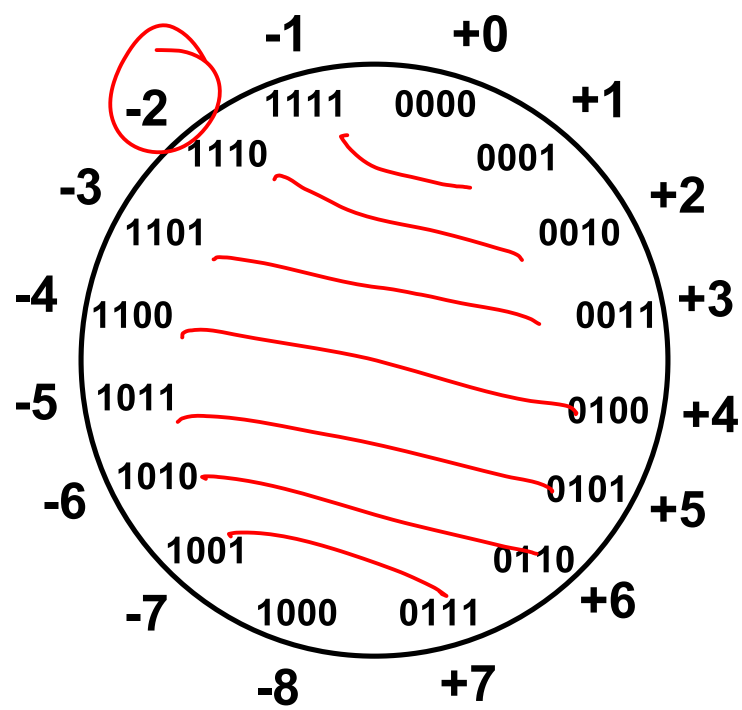
1101

1110 = -2

- Negate $(1110)_2$ (-2)

0001

0010 = +2



Addition in Two's Complement

$$\begin{array}{r r r r r l} & 0 & 0 & 1 & 0 & (+2) \\ + & 0 & 1 & 0 & 0 & (+4) \\ \hline & 0 & 1 & 1 & 0 & = 6 \end{array}$$

$$\begin{array}{r r r r r r l} & & 1 & & & & \\ & & 1 & 1 & 1 & 0 & (-2) \\ + & & 1 & 1 & 0 & 0 & (-4) \\ \hline & 1 & 1 & 0 & 1 & 0 & = -6 \end{array}$$

$$\begin{array}{r r r r r l} & 0 & 0 & 1 & 0 & (+2) \\ + & 1 & 1 & 0 & 0 & (-4) \\ \hline & 1 & 1 & 1 & 0 & -2 \end{array}$$

$$\begin{array}{r r r r r r l} & & 1 & & & & \\ & & 1 & 1 & 1 & 0 & (-2) \\ + & & 0 & 1 & 0 & 0 & (+4) \\ \hline & 1 & 0 & 0 & 1 & 0 & = +2 \end{array}$$

Subtraction in Two's Complement

- $A - B = A + (-B) = A + \overline{B} + 1$

- 0010 - 0110

  $2 - 6 = -4$

  $$\begin{array}{r} 0010 \\ -\ 0110 \end{array} \rightarrow \begin{array}{r} 0010 \\ 1001 \\ +1 \\ \hline 1100 \end{array} = -4$$

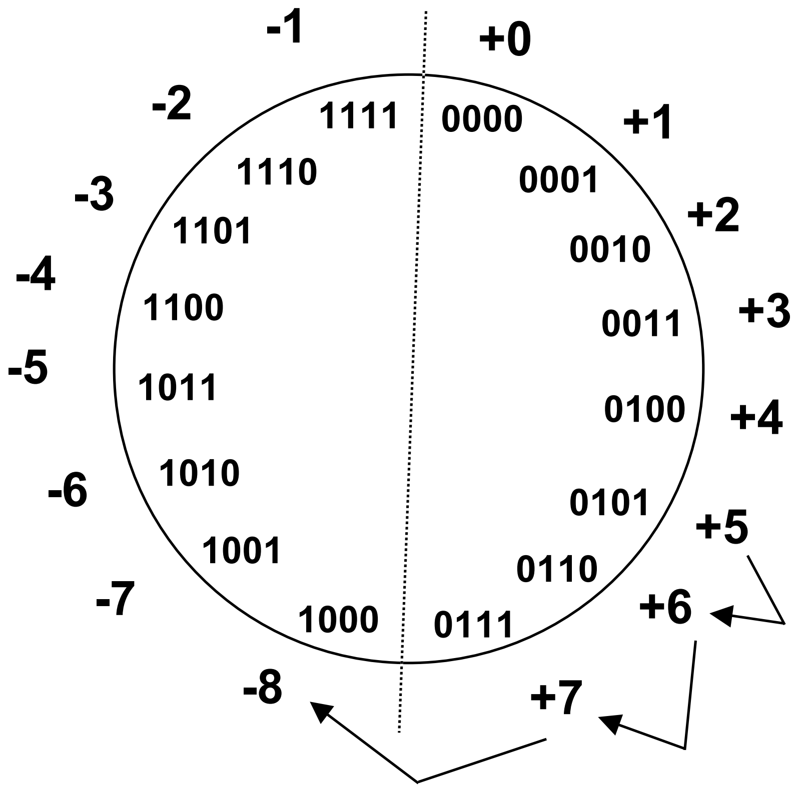
- 1011 - 1001

- 1011 - 0001

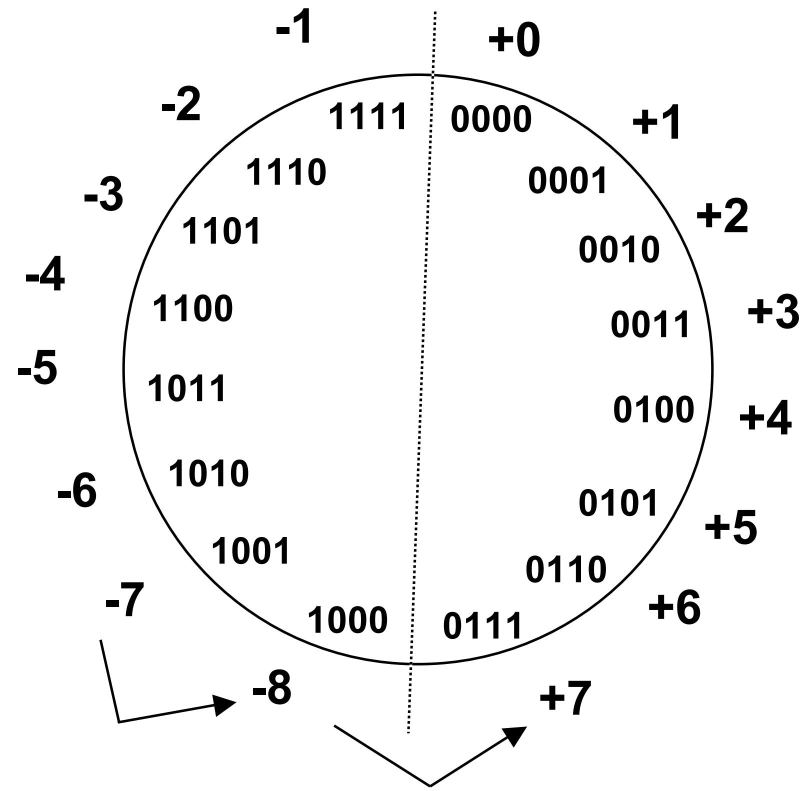
Overflows in Two's Complement

Add two positive numbers to get a negative number

or two negative numbers to get a positive number



5 + 3 = -8



-7 - 2 = +7

Overflow Detection in Two's Complement

		1 1 1
5		0 1 0 1
3		0 0 1 1
-8	0	1 0 0 0

Overflow

		0		1 1 1 1
-7		1 0 0 1	−1	1 1 1 1
-2		1 1 1 0	+ +1	0 0 0 1
7	1	0 1 1 1	1	0 0 0 0

Overflow

		0
5		0 1 0 1
2		0 0 1 0
7	0	0 1 1 1

No overflow

		1 1 1
-3		1 1 0 1
-5		1 0 1 1
-8	1	1 0 0 0

No overflow

= XOR bit position

Overflow when carry in to sign bit position does not equal carry out

Converting Decimal to Two's Complement

- Convert absolute value to binary, then negate if necessary
- Convert $(-9)_{10}$ to 6-bit Two's Complement

$$|-9_{10}| = 9_{10} = 001001_2 = \begin{array}{r} 110110 \\ +\quad 1 \\ \hline (110111)_2 \end{array}$$

- Convert $(9)_{10}$ to 6-bit Two's Complement

Converting Two's Complement to Decimal

- If Positive, convert as normal;
  If Negative, negate then convert.

- Convert $(11010)_2$ to Decimal

$$\begin{array}{r} 00101 \\ + \quad 1 \\ \hline (00110)_2 \end{array} \rightarrow -6_{10}$$

- Convert $(01011)_2$ to Decimal

$(11)_{10}$

Sign Extension

- To convert from N-bit to M-bit Two's Complement (N>M), simply duplicate sign bit:

- Convert $(1011)_2$ to 8-bit Two's Complement

$1011_2 \rightarrow 1111\,1011_2$

$1000_2 = 8_{10}$

$\Downarrow$

$0000\,1000 = 8_{10}$

- Convert $(0010)_2$ to 8-bit Two's Complement

$0010_2 \rightarrow 0000\ 0010_2$